STEFAN-BOLTZMANN LAW

The Stefan-Boltzmann law states that the total energy radiated per unit surface area of a black body across all wavelengths per unit time, $j^\star$, is directly proportional to the fourth power of the black body's thermodynamic temperature T:

$$j^\star = \sigma T^4. \tag{1}$$

where σ is the Stefan-Boltzmann constant derived from other konwn constants.

Herein, we are interested to derive this T^4 relationship using classical thermodynamics. We start form the **Maxwell stress tensor** of the classical electrodynamics that the radiation pressure p is related to the internal energy density u as

$$p = \frac{u}{3} \tag{2}$$

From the fundamental thermodynamic relation

$$dU = TdS - pdV \tag{3}$$

we obtain the following expression, after dividing by dV and fixing T :

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p \tag{4}$$

The last equality comes from the following Maxwell relation:

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V. \tag{5}$$

From the definition of energy density it follows that

$$U = uV \tag{6}$$

where the energy density of radiation *only* depends on the temperature, therefore

$$\left(\frac{\partial U}{\partial V}\right)_T = u\left(\frac{\partial V}{\partial V}\right)_T = u. \tag{7}$$

Now, the equality

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p, \tag{8}$$

after substitution of $\left(\frac{\partial U}{\partial V}\right)_T$ and p for the corresponding expressions, can be written as

$$u = \frac{T}{3}\left(\frac{\partial u}{\partial T}\right)_V - \frac{u}{3}. \tag{9}$$

Since the partial derivative $\left(\frac{\partial u}{\partial T}\right)_V$ can be expressed as a relationship between only u and T (if one isolates it on one side of the equality), the partial derivative can be replaced by the ordinary derivative. After separating the differentials the equality becomes

$$\frac{du}{4u} = \frac{dT}{T}, \tag{10}$$

which leads immediately to $u = aT^4$, with a as some constant of integration.

THE END