

VECTORS

1. INTRODUCTION

A **vector** is a quantity that has both *magnitude* and *direction*. This is in contrast to a (more familiar) **scalar**, which has only magnitude. Vectors are important as many quantities are direction-dependent; examples of vectors which you may have encountered include force, velocity and electric field. Here we will follow the physical approach and leave the mathematical definitions to the mathematicians. We will first deal with vectors in the *Euclidean* 3D space which allows easy visualisation, although we note that the notion of vectors extend to higher dimensions.

2. NOTATION

In the Euclidean 3D space, a vector can be represented as an arrow having a start point and an end point. Figure 1 shows 3 vectors. To distinguish vectors from scalars, we put vector quantities in boldface when printing and underline it when writing. Vectors represented by a single small letter can thus be written as **a** and **b** in printed work, or as a and b in written work (this is the convention we adopt here). Sometimes, a tilde is placed underneath the vector or an overhead arrow is used (e.g. \tilde{a} or \vec{a}). If both the start and the end points of the vector are given, then the vector can be represented as \overrightarrow{AB} , where the overhead arrow directs from the start point, A , to the end point, B .

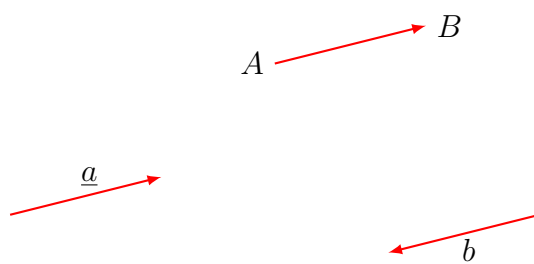


FIGURE 1. Geometrical representation of vectors.

A three-dimensional vector can be written as an *ordered* list of 3 elements; we can represent vector $\mathbf{a} = \underline{a}$ as (a_1, a_2, a_3) where $a_i, i = 1, 2, 3$ are the components of the vector a. We can generalise this to an n -dimensional vector, which we represent as an n -tuple, that is, as an ordered list of n elements, i.e., (a_1, a_2, \dots, a_n) . Note that these vector components are unique for a given vector with an associated *basis*.

2.1. Bases and Cartesian Coordinate System.

We introduce the concept of a **basis**. In n -dimension, the basis is a collection of n *linearly independent* vectors. A set of vectors are linearly independent when no one vector can be written as a linear combination of other vectors. These linearly independent vectors are said to *span* the n -dimensional space. This means that any given n -dimensional vector can be written as a linear combination of these basis vectors. Specifically in three-dimension, we can write any vector \underline{r} as

$$\underline{r} = \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} \quad (1)$$

where the three non-coplanar vectors $\{\underline{a}, \underline{b}, \underline{c}\}$ constitutes a basis. Note that the choice of basis is not unique; any three non-coplanar vectors may be chosen.

In the Cartesian coordinate system (Figure 2), we choose **orthonormal** basis and denote these as $\{\underline{i}, \underline{j}, \underline{k}\}$, each in the x -, y - and z -direction respectively. They are *orthogonal* to one another, i.e.,

$$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0 \quad (2)$$

and *normalized*, that is they have unit length, viz.,

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1. \quad (3)$$

In the Cartesian coordinate system, therefore, any vector \underline{a} can then be written as $\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$, where a_x , a_y and a_z are scalars. In fact, you should have noticed that in the above, we have written the vector \underline{a} as a 3-tuple, (a_x, a_y, a_z) . This is strictly an abuse of notation as we have implicitly assumed that a basis has been adopted and subsequently suppressed in that notation.

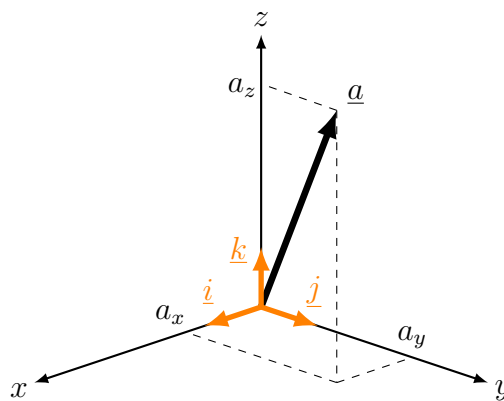


FIGURE 2. Cartesian coordinate system showing a vector and its component along the basis vectors.

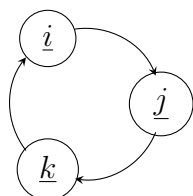
By convention, we also choose the Cartesian coordinate to be *right-handed* coordinate system. This means that if we use the *right-hand rule* going from x -axis to y -axis, our

thumb will point to the z -axis. Following this definition, the basis vectors obey the following cross product (see later) relations:

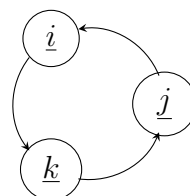
$$\underline{i} \times \underline{j} = \underline{k} \quad \underline{j} \times \underline{k} = \underline{i} \quad \underline{k} \times \underline{i} = \underline{j} \quad (\text{cyclic}) \quad (4)$$

$$\underline{j} \times \underline{i} = -\underline{k} \quad \underline{k} \times \underline{j} = -\underline{i} \quad \underline{i} \times \underline{k} = -\underline{j} \quad (\text{anti-cyclic}) \quad (5)$$

which can be remembered more easily as



cyclic relations give positive sign



anti-cyclic relations give negative sign

2.2. Magnitude, Zero Vectors and Unit Vectors.

We can now define various vectors in terms of their components in the Cartesian basis. The magnitude of a given vector, \mathbf{a} , is a scalar and is denoted by $a = |\mathbf{a}| = |\underline{a}|$. For a three-dimensional vector, $\underline{a} = (a_1, a_2, a_3)$, we have

$$a = \sqrt{(a_1^2 + a_2^2 + a_3^2)} \quad (6)$$

and more generally, for an n -dimensional vector, its magnitude is given by

$$a = \sqrt{(a_1^2 + a_2^2 + \cdots + a_n^2)} = \sqrt{\sum_{i=1}^n a_i^2} \quad (7)$$

where \sum is the summation sign. Note that $a \geq 0$ since the length of a vector cannot be negative; the equality holds when the components $a_i = 0$ for all $i = 1, 2, \dots, n$, i.e., when the vector is a zero vector. Also note that Equations (6) and (7) are true for coordinate systems with orthonormal basis.

Example 2.1. Find the magnitude of the following vectors: (i) $\underline{a} = (1, 2, 3, 4)$ and (ii) $\mathbf{p} = (2, -3, 9)$.

Solution 2.1. (i) The magnitude of $\underline{a} = (1, 2, 3, 4)$ is given by $|\underline{a}| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$.

(ii) The magnitude of $\mathbf{p} = (2, -3, 9)$ is given by $|\mathbf{p}| = \sqrt{2^2 + (-3)^2 + 9^2} = \sqrt{94}$.

A **zero vector** has no magnitude and can take any direction. This is denoted by $\mathbf{0}$ in printing or by $\underline{0}$ in writing to distinguish it from the scalar zero.

A **unit vector** is a vector that has a magnitude of one (unity) and a specified direction. It is denoted with a caret (or hat) on top of the vector. For example, $\hat{\mathbf{a}}$ is a unit vector

with unity magnitude and takes the direction of the vector \mathbf{a} . A unit vector can be obtained from any vector by dividing the vector by its magnitude, i.e.,

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\underline{u}}{|\underline{u}|} \quad (8)$$

Example 2.2. Find the unit vector of the following vectors: (i) $\underline{a} = (1, 2, 3, 4)$ and (ii) $\underline{p} = (2, -3, 9)$.

Solution 2.2. We have the magnitudes of these vectors from Example 2.1. Therefore,

(i) the unit vector of $\underline{a} = (1, 2, 3, 4)$ is given by $\hat{\underline{a}} = \frac{1}{\sqrt{30}}(1, 2, 3, 4)$.

(ii) the unit vector of $\underline{p} = (2, -3, 9)$ is given by $\hat{\underline{p}} = \frac{1}{\sqrt{94}}(2, -3, 9)$.

Two vectors are equal when they have the same magnitude *and* direction. In Figure 1, we have $\vec{a} = \vec{AB}$. This implies that a vector only depends on the *relative* positions of its start and end points. Although vector \vec{b} has the same magnitude as the other two, it has a different direction, thus it is a different vector.

3. VECTOR ALGEBRA

3.1. Addition of Vectors.

Figure 3 shows the addition of two vectors \underline{a} and \underline{b} . We can think of \underline{a} as the *displacement* between point A and point B , i.e., $\underline{a} = \vec{AB}$. A displacement is a vector as it has both magnitude and direction. If we want to go from point A to point C , we could go directly or we could go via B . Vectorially, we represent this as $\vec{AC} = \vec{AB} + \vec{BC}$. We can simply write this as

$$\underline{c} = \underline{a} + \underline{b}, \quad (9)$$

where $\underline{a} = \vec{AB}$ and so on. This illustrates the addition of vectors. We obtain the resultant vector, \underline{c} , by joining the head of a vector, \underline{a} , to the tail of another vector, \underline{b} , at a common point B .

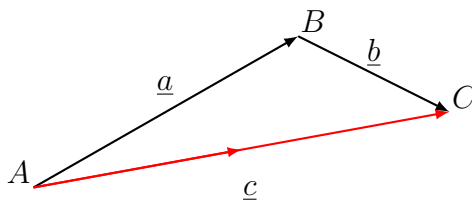


FIGURE 3. Vector addition by joining the adding vectors from head to tail to get the resultant vector.

We can therefore *define* the addition of vectors by the composition of their associated displacements as given in Equation (9). The resultant vector is obtained by the operation on the components of the adding vectors.

Example 3.1. Find the resultant vector of the following vector additions: (i) $\underline{a} + \underline{b}$ where $\underline{a} = (1, 2, 3, 4)$ and $\underline{b} = (5, 6, 7, 8)$ and (ii) $\underline{p} + \underline{q}$ where $\underline{p} = (2, -3, 9)$ and $\underline{q} = (-2, 3, -9)$.

Solution 3.1. We add the vectors component-wise to obtain

$$(i) \underline{a} + \underline{b} = (1, 2, 3, 4) + (5, 6, 7, 8) = (6, 8, 10, 12)$$

$$(ii) \underline{p} + \underline{q} = (2, -3, 9) + (-2, 3, -9) = (0, 0, 0) = \underline{\mathbf{0}} = \underline{\mathbf{0}}$$

3.2. Subtraction of Vectors.

Given a vector \underline{a} representing the displacement from point A to point B , i.e., $\underline{a} = \overrightarrow{AB}$, we can define another vector $\underline{b} = -\underline{a}$ that goes from point B to point A . Vector \underline{b} has the same magnitude as vector \underline{a} , but has the opposite direction as vector \underline{a} ; this is indicated by the negative sign.

We can thus think of the subtraction of a vector from another as the addition of the same vector in the opposite direction. In Figure 4, to subtract vector \underline{b} from vector \underline{a} , we simply add vector $-\underline{b}$ to it, i.e.,

$$\underline{a} - \underline{b} = \underline{a} + (-\underline{b}) = \underline{c}. \quad (10)$$

We call vector $-\underline{b}$ the **additive inverse** of vector \underline{b} since they add up to the zero vector, i.e., $\underline{b} + (-\underline{b}) = \underline{\mathbf{0}}$.

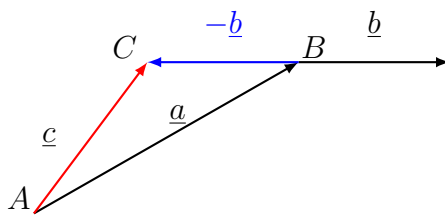


FIGURE 4. Vector subtraction by the addition of the additive inverse.

3.3. Multiplication by a scalar.

We can multiply a vector by a (real) scalar to form another vector. Suppose we multiply a vector \underline{a} by a scalar λ , we obtain a new vector denoted by $\lambda\underline{a}$. The resultant vector has the magnitude of $|\lambda\underline{a}|$. Furthermore, if λ is positive, then the resultant vector has the same direction as the original vector. If, on the other hand, λ is negative, then the resultant vector has the opposite direction as the original vector. If instead, vector \underline{a} is multiplied by 0, then the result is a *vector* with zero magnitude. In other words, any non-zero vector, when multiplied by the scalar 0, gives the zero vector.

Example 3.2. Find the resultant vector when the following vectors are multiplied by a scalar: (i) $\lambda \underline{a}$ where $\underline{a} = (1, 2, 3, 4)$ and $\lambda = -2$ and (ii) $\lambda \underline{p}$ where $\underline{p} = (2, -3, 9)$ and $\lambda = 0$.

Solution 3.2. We multiply each component of the vector to obtain

$$(i) \lambda \underline{a} = -2(1, 2, 3, 4) = (-2, -4, -6, -8)$$

$$(ii) \lambda \underline{p} = 0(2, -3, 9) = (0, 0, 0) = \underline{\mathbf{0}} = \underline{\mathbf{0}}$$

3.4. Commutativity and Associativity of vector additions and Distributivity of scalar multiplications.

Since vectors depend only on their relative positions, we can translate a vector without changing it. Consider the vector addition in Figure 5. We can go from point A to point C by going via point B , in which case, we obtain the resultant vector $\underline{c} = \underline{a} + \underline{b}$. Since vectors are invariant to translation, we note that vector $\overrightarrow{AB} = \underline{a} = \overrightarrow{B'C}$ and that vector $\overrightarrow{BC} = \underline{b} = \overrightarrow{AB'}$. Therefore, we can go from point A to point C via point B' , thus obtaining $\underline{c} = \underline{b} + \underline{a}$. Therefore, we have shown that vector additions are **commutative**, i.e.,

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}. \quad (11)$$

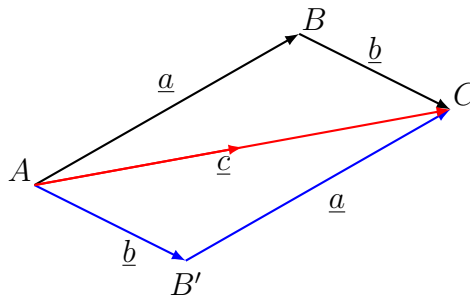


FIGURE 5. Commutativity of vector addition by considering a parallelogram.

Vector additions are also **associative**. This is the same as for the addition of scalars. Vectorially, we have

$$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c} \equiv \underline{a} + \underline{b} + \underline{c} \quad (12)$$

We can show this easily by considering the geometry of a parallelepiped (Figure 6). Here, we could go from point A to point E by following any three edges of the parallelepiped and adding the three vectors \underline{a} , \underline{b} and \underline{c} in any order. Algebraically, the commutativity and associativity of vector additions follow directly from the commutativity and associativity of the components of the vectors.

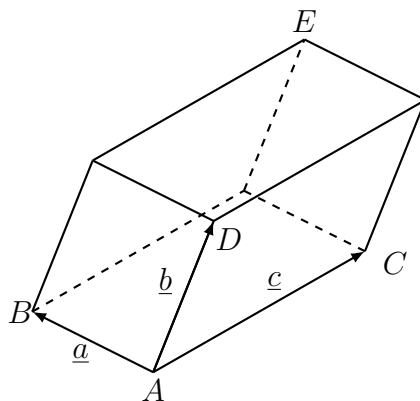


FIGURE 6. Associativity of vector addition. We can go from point A to point E following any three edges of the given parallelepiped, thus adding the three vectors in any order.

Vector multiplication by a scalar is **distributive** over addition,

$$\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}, \quad (13)$$

which follows directly from the component-wise manipulation of the vectors.

4. SCALAR PRODUCT

We have seen the multiplication of a vector by a scalar as well as the addition and subtraction of vectors. Here we define an operation that combines two given vectors to return a scalar. The **scalar product** or the **dot product** between two vectors \underline{a} and \underline{b} is given by

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta \quad (14)$$

where θ is the angle formed by the two vectors (Figure 7).

We can interpret the scalar product as the product of the length of vector \underline{a} ($|\underline{a}|$) and the length of the projection of the vector \underline{b} on vector \underline{a} ($|\underline{b}| \cos \theta$).

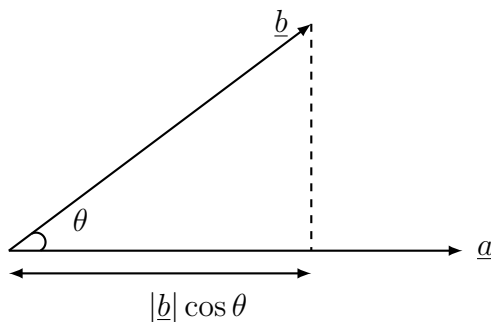


FIGURE 7. Definition of scalar product.

Algebraically, in terms of the components of the two vectors, we have

$$\underline{a} \cdot \underline{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^3 a_ib_i \quad (15)$$

in three-dimension. More generally, for two n -dimensional vectors, we have,

$$\underline{a} \cdot \underline{b} = (a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = a_1b_1 + a_2b_2 + \dots + a_nb_n = \sum_{i=1}^n a_ib_i. \quad (16)$$

where we have assumed that the *orthonormal* basis has been used.

Note that the result of a scalar product of two vectors is a number, thus the name *scalar* product. The dot notation of the scalar product gives rise to the alternative name of the *dot* product.

Example 4.1. Find the scalar product of the following vectors: (i) $\underline{a} \cdot \underline{b}$ where $\underline{a} = (1, 2, 3, 4)$ and $\underline{b} = (5, 6, 7, 8)$, (ii) $\underline{p} \cdot \underline{q}$ where $\underline{p} = (2, -3, 9)$ and $\underline{q} = (-2, 3, -9)$, (iii) $\underline{a} \cdot \underline{a}$ and (iv) $\underline{a} \cdot \underline{b}$ given that the vectors are perpendicular to each other.

Solution 4.1. Using the general formula Equation (16), we have

$$(i) \underline{a} \cdot \underline{b} = (1, 2, 3, 4) \cdot (5, 6, 7, 8) = 1 \times 5 + 2 \times 6 + 3 \times 7 + 4 \times 8 = 70$$

$$(ii) \underline{p} \cdot \underline{q} = (2, -3, 9) \cdot (-2, 3, -9) = 2 \times (-2) + (-3) \times 3 + 9 \times (-9) = -94$$

$$(iii) \underline{a} \cdot \underline{a} = |\underline{a}||\underline{a}|\cos 0 = |\underline{a}|^2$$

$$(iv) \text{ Since the vectors are perpendicular, the angle between them are } 90^\circ = \pi/2, \text{ thus, we have } \underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos(\pi/2) = 0$$

Note that the results are scalars, as expected.

The converse of part (iv) in the above example is that, if $\underline{a} \cdot \underline{b} = 0$, then either (1) vectors \underline{a} and \underline{b} are perpendicular, in which case they are said to be **orthogonal**, or (2) one (or both) of these two vectors is (are) zero vector(s).

Example 4.2. A vector \underline{b} can be resolved into two component vectors, one parallel to and the other perpendicular to a given unit vector \hat{u} (Figure 8). The part that is parallel to \hat{u} is given by $(\underline{b} \cdot \hat{u})\hat{u}$ and the remaining part that is perpendicular to \hat{u} is given by $\underline{b} - (\underline{b} \cdot \hat{u})\hat{u}$ since $\underline{b} \cdot \hat{u} = |\underline{b}||\hat{u}|\cos \theta = |\underline{b}|\cos \theta$ where we have used the fact that the unit vector has a magnitude of unity, $|\hat{u}| = 1$.

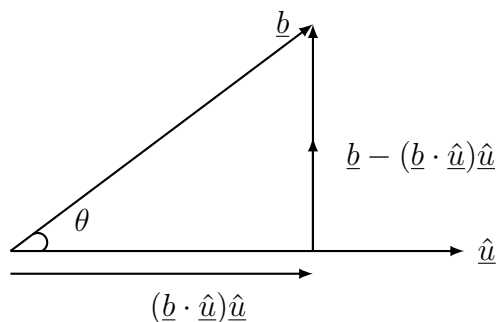


FIGURE 8. One can resolve a given vector into two component vectors that are parallel and perpendicular to a given unit vector.

Scalar products are **commutative**, that is

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \quad (17)$$

It is also **distributive** over addition, that is

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad (18)$$

Additionally, when a scalar product is multiplied by a scalar, we have

$$\lambda(\underline{a} \cdot \underline{b}) = (\lambda\underline{a}) \cdot \underline{b} = \underline{a} \cdot (\lambda\underline{b}) \quad (19)$$

The above three equations follow directly by considering the operation of the components of the vectors. For Equation (18), we can consider a geometrical proof. Consider Figure 9. The projection of vector $(\underline{b} + \underline{c})$ onto vector \underline{a} is given by the the scalar product $(\underline{b} + \underline{c}) \cdot \underline{a} = |(\underline{b} + \underline{c})||\underline{a}| \cos \theta = |\underline{a}||\overrightarrow{ON}| = \underline{a} \cdot (\underline{b} + \underline{c})$. Therefore, we have

$$\underline{a} \cdot (\underline{b} + \underline{c}) = |\underline{a}||\overrightarrow{ON}| = |\underline{a}|(|\overrightarrow{OM}| + |\overrightarrow{MN}|) = |\underline{a}||\overrightarrow{OM}| + |\underline{a}||\overrightarrow{MN}| = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad (20)$$

□

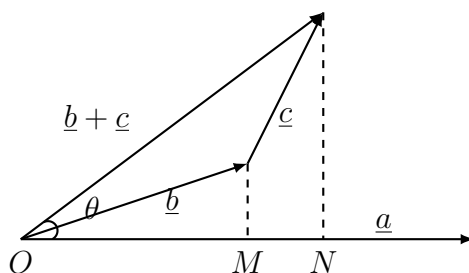


FIGURE 9. Proof of distributivity of scalar product using geometrical argument.

A nice application of the scalar product and its associated properties is the derivation of the *cosine rule*. Consider Figure 10. We have

$$\begin{aligned}
 |c|^2 &= |\underline{a} + \underline{b}|^2 \\
 &= (\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) \\
 &= \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b} \\
 &= \underline{a} \cdot \underline{a} + 2\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b} \\
 &= |\underline{a}|^2 + |\underline{b}|^2 + 2|\underline{a}||\underline{b}|\cos(\pi - \theta) \\
 &= |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos(\theta)
 \end{aligned} \tag{21}$$

□

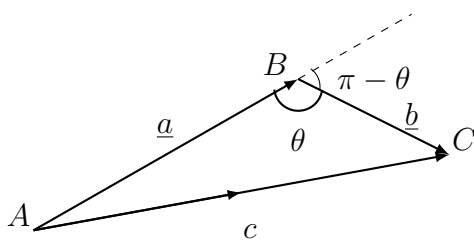
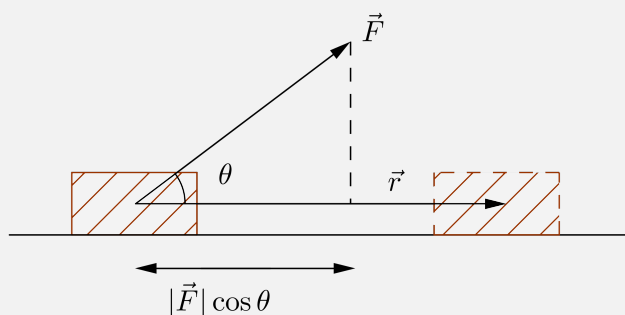


FIGURE 10. Vector addition by joining the adding vectors from head to tail to get the resultant vector.

Physical Illustration 4.1. The mechanical work is the dot product of the force vector and the displacement vector.

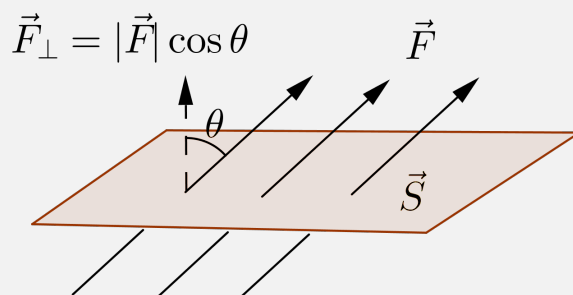
$$W = \vec{F} \cdot \vec{r} \tag{22}$$



The work done is the product of the force and the distance in the direction of the force. From the diagram, we note that the magnitude of the force \vec{F} in the direction of the displacement vector is given by $|\vec{F}| \cos \theta$, thus the work done in moving the block is $(|\vec{F}| \cos \theta)(|r|) = |\vec{F}||r| \cos \theta = \vec{F} \cdot \vec{r}$.

Physical Illustration 4.2. The flux of a vector field is the dot product of the field vector and the vector surface area.

$$\Phi = \vec{F} \cdot \vec{S} \quad (23)$$



The **magnetic flux**, Φ_B , measures the total magnetic field \vec{B} passing through a given surface area. This is given by the product of the normal component of the magnetic field and the surface area that the field passes through. Thus, $\Phi_B = (|\vec{B}| \cos \theta)(\vec{S}) = |\vec{B}|\vec{S} \cos \theta = \vec{B} \cdot \vec{S}$.

Similarly, the **electric flux**, Φ_E , measures the total electric field \vec{E} passing through a given surface area. This is given by the product of the normal component of the electric field and the surface area that the field passes through. Thus, $\Phi_E = (|\vec{E}| \cos \theta)(\vec{S}) = |\vec{E}|\vec{S} \cos \theta = \vec{E} \cdot \vec{S}$.

5. VECTOR PRODUCT

Having seen the operation between two vectors that returns a scalar as a result, we introduce here another operation that returns a vector from the composition of two vectors. The **vector product** or the **cross product** between two vectors \underline{a} and \underline{b} is defined by

$$\underline{a} \wedge \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{n} \quad (24)$$

where the angle θ is the smaller angle between vectors \underline{a} and \underline{b} ; \hat{n} is the unit vector perpendicular to both vectors \underline{a} and \underline{b} in the direction formed by the *right-hand rule* from \underline{a} to \underline{b} (Figure 11). We note that the result of this operation is a vector, thus the name *vector product*. Sometimes, we denote the vector product as $\underline{a} \times \underline{b}$, which gives rise to the alternative name of the *cross product*.

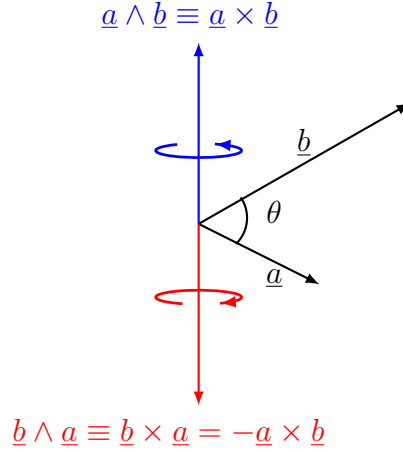


FIGURE 11. Definition of vector product using the right-hand rule.

Note that if instead we have $\underline{b} \times \underline{a}$, then the right hand rule would give the resultant vector that is in the opposite direction to that given by $\underline{a} \times \underline{b}$, thus, we have

$$\underline{b} \times \underline{a} = -\underline{a} \times \underline{b} \quad (25)$$

We can easily find the cross product of two given vectors in Cartesian coordinate system. Suppose vector $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and vector $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$, then their cross product can be written as a **determinant**,

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (26)$$

which is a 3×3 determinant that expands out to be a sum of 2×2 determinants that further simplifies to give a vector, viz.,

$$\begin{aligned} \underline{a} \times \underline{b} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \underline{i}(a_2b_3 - b_2a_3) - \underline{j}(a_1b_3 - b_1a_3) + \underline{k}(a_1b_2 - a_2b_1) \end{aligned} \quad (27)$$

One property of determinants is that they change sign when any two rows or two columns are interchanged. If we write $\underline{b} \times \underline{a}$ as a determinant, then we have

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = - \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\underline{a} \times \underline{b}$$

which is what we expect from geometrical arguments using the right-hand rule.

When a cross product is multiplied by a scalar, λ , we have

$$\lambda(\underline{a} \times \underline{b}) = (\lambda\underline{a}) \times \underline{b} = \underline{a} \times (\lambda\underline{b}). \quad (28)$$

This can easily be proven using the definition of vector product (Equation 24) or using the property of determinant that

$$(\lambda\underline{a}) \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \lambda \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ \lambda b_1 & \lambda b_2 & \lambda b_3 \end{vmatrix} = \underline{a} \times (\lambda\underline{b}) \quad (29)$$

where we have used the property of a determinant to factor out (or multiply in) a common factor from (or to) any row. Thinking geometrically using the right hand rule and noting that multiplying a vector by a scalar either preserves its direction or reverses it would allow us to reach the same conclusion.

Vector product is **distributive** over addition, that is

$$\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}. \quad (30)$$

To prove this, we introduce the concept of a vector area.

5.1. Vector Area.

Define **vector area** \underline{S} as a *directed* surface area with magnitude $|\underline{S}|$ and direction perpendicular to the surface. The ambiguity of the direction is fixed by applying the right hand rule around the perimeter of the area such that the thumb points to the direction of the vector area. We call this the outwards direction (Figure 12) and denote it by a unit vector \hat{n} . We can also project this vector area onto the xy -plane and obtain a projected area given by $\underline{S} \cdot \underline{k}$ where \underline{k} is the unit vector in the z -direction. This is important when we consider surface integrals later.

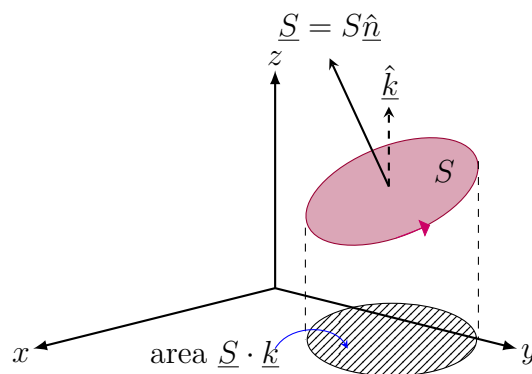


FIGURE 12. Definition of vector area as a *directed* surface with area S in the direction of unit normal vector \hat{n} . We also show its projection onto the xy -plane.

For a general surface, the directed surface area is only dependent on its rim. We can think of this as dividing the surface into small vector area elements $\delta \underline{S}$ such that those falling outside the rim have opposite unit normal vectors and cancel (Figure 13).

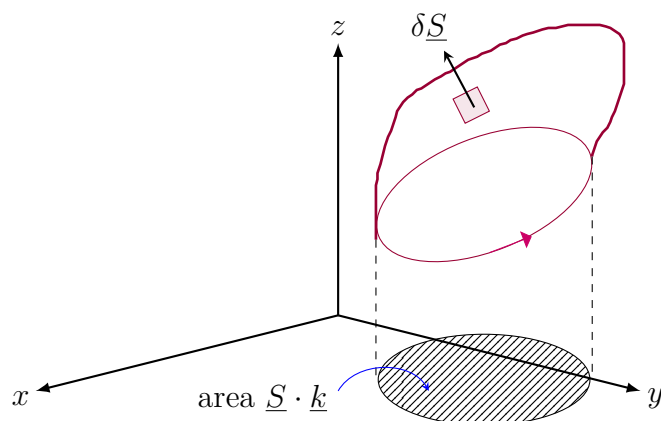


FIGURE 13. A general vector area with its projected area in the xy -plane. Note that the projected area depends only on the rim of the general surface.

In the limit when we divide the vector area into infinite pieces of vector area elements, we integrate to obtain the total vector area of the general surface,

$$\underline{S} = \int d\underline{S}. \quad (31)$$

By the argument above, we also note that the vector area of a *closed* surface is zero, i.e.,

$$\oint d\underline{S} = \underline{S} = \underline{0}. \quad (32)$$

From the definition of vector product (Equation 24), we know that $\underline{a} \times \underline{b}$ gives the directed area of the parallelogram formed by the two vectors \underline{a} and \underline{b} where the direction of the area is normal to both vectors \underline{a} and \underline{b} and is given by the right hand rule.

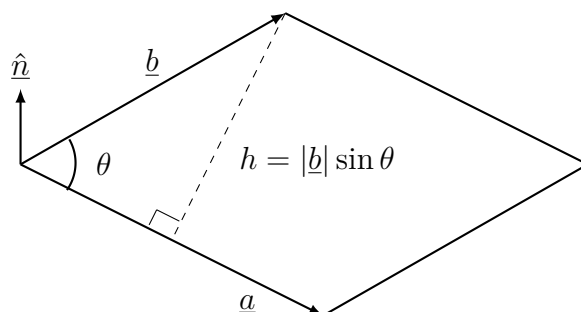


FIGURE 14. The (directed) area of a parallelogram is given by the cross product of the two vectors forming its sides. The direction of the resultant area follows the right-hand rule.

We are now in a position to prove the distributive law of vector product. Consider the closed surface area in Figure 15. Since the areas are closed, the vector areas sum to zero. When adding the vector areas, the vector area from the top and that from the bottom of the prism cancel as they have opposite outward normal vectors. This implies that the sum of the vector areas from the three sides of the prism gives zero contribution:

$$\begin{aligned}\underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{a} \times (\underline{b} + \underline{c}) &= \underline{0} \\ \implies \underline{a} \times (\underline{b} + \underline{c}) &= -\underline{b} \times \underline{a} - \underline{c} \times \underline{a} \\ \implies \underline{a} \times (\underline{b} + \underline{c}) &= \underline{a} \times \underline{b} + \underline{a} \times \underline{c}\end{aligned}$$

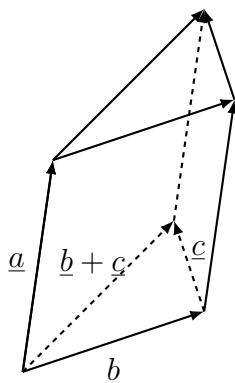
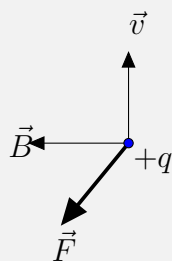


FIGURE 15. Proof of the distributivity of vector product by considering the sum of the vector areas of a triangular prism.

This completes our proof of the distributivity of vector product.

Physical Illustration 5.1. The **magnetic force**, \vec{F} , experienced by a charge $+q$ moving with velocity \vec{v} in a magnetic field \vec{B} , is given by the cross product of $q\vec{v}$ and \vec{B} .

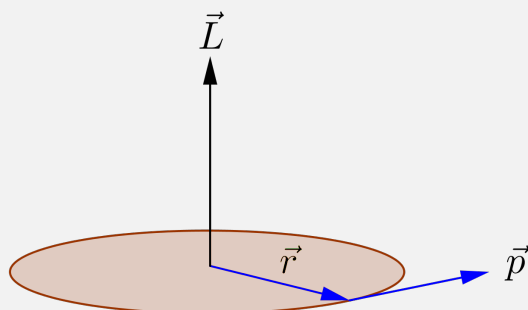
$$\vec{F} = q\vec{v} \times \vec{B} \quad (33)$$



Again, by definition, the magnetic force is in a direction that is perpendicular to both the velocity vector of the moving charge and the magnetic field. The direction is given by the right hand rule.

Physical Illustration 5.2. The **angular momentum** of a particle, \vec{L} , is given by the cross product of its position vector \vec{r} and momentum vector \vec{p} .

$$\vec{L} = \vec{r} \times \vec{p} \quad (34)$$



The angular momentum of a particle rotating about an origin is given by $\vec{L} = I\vec{\omega}$ where

$$I = mr^2 \quad (35)$$

is the **moment of inertia** of the particle with mass m and

$$\omega = \frac{\vec{r} \times \vec{v}}{r^2} \quad (36)$$

is the angular velocity of the particle about the origin.

Substituting these equations, we have

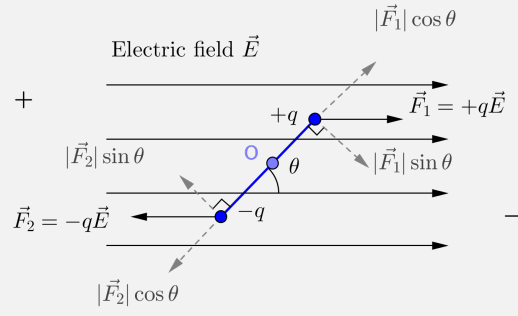
$$\vec{L} = I\vec{\omega} = (mr^2) \left(\frac{\vec{r} \times \vec{v}}{r^2} \right) = m(\vec{r} \times \vec{v}) = \vec{r} \times (m\vec{v}) = \vec{r} \times \vec{p} \quad (37)$$

Note that, by definition, the angular momentum of the particle is in a direction that is perpendicular to both its position and velocity vectors.

Physical Illustration 5.3. The **torque** on an electric dipole produced by the electric field that the dipole is in is given by the cross product of the dipole moment \vec{p} and the electric field \vec{E} .

$$\vec{\tau} = \vec{p} \times \vec{E} \quad (38)$$

Suppose we have a neutral diatomic molecule with dipole moment $\vec{p} = q\vec{r}$ where q is the charge and \vec{r} is the displacement vector from the positive end to the negative end. When placed in an electric field (as shown in the figure), each of the two charges experiences a force given by $\vec{F} = q\vec{E}$.



We can resolve each of the forces into parallel and perpendicular components. We note that the forces causing the stretching of the molecule (i.e. $|\vec{F}_1| \cos \theta$ and $|\vec{F}_2| \cos \theta$; along the dipole) cancel each other as they have the same magnitude but are in opposite directions.

The components of the force perpendicular to the dipole cause rotational motion (i.e. a torque) of the molecule about its centre of mass and the magnitude of this torque is given by

$$\begin{aligned}
 |\vec{\tau}| &= \left(|\vec{F}_1| \sin \theta \right) \left(\frac{r}{2} \right) + \left(|\vec{F}_2| \sin \theta \right) \left(\frac{r}{2} \right) \\
 &= |\vec{F}| |\vec{r}| \sin \theta \\
 &= |\vec{F}| |\vec{r}| |\hat{n}| \sin \theta \\
 &= |q\vec{E}| |\vec{r}| |\hat{n}| \sin \theta \\
 &= |q\vec{r}| |\vec{E}| |\hat{n}| \sin \theta \\
 &= |\vec{p} \times \vec{E}|
 \end{aligned}$$

where we have noted that $|\vec{F}_1| = |\vec{F}_2| = |\vec{F}|$ and have introduced a unit normal vector \hat{n} that is perpendicular to both \vec{F} and \vec{r} ($|\hat{n}| = 1$).

In vector notation, we then obtain Equation (38). The direction of the resultant torque is fixed using the right hand rule.

Note that, by definition, the torque produced by the electric field on the dipole is in a direction that is perpendicular to both the dipole moment and the electric field.

6. SCALAR AND VECTOR TRIPLE PRODUCTS

6.1. Scalar Triple Product.

We define the **scalar triple product** of *three* vectors, \underline{a} , \underline{b} and \underline{c} , as

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}) \quad (39)$$

The result of this composition is a *scalar* that equals to the volume of the parallelepiped formed by the given three vectors \underline{a} , \underline{b} and \underline{c} . Note that the scalar triple product is non-zero if the three vectors are non-coplanar, that is the three vectors do not all lie on the same plane. This is equivalent to the condition that the three vectors are linearly independent.

We can prove that the scalar product gives the volume of the parallelepiped by considering Figure 6. The vector product $\underline{b} \times \underline{c}$ gives the (directed) base area of the parallelogram and its subsequent dot product with vector \underline{a} gives the height of the parallelepiped from the parallelogram base area. Thus the resultant scalar gives its volume. The volume of the parallelepiped can also be formed from different permutations of the three vectors, giving rise to the *cyclic* relations of the scalar triple products, viz.,

$$[\underline{a}, \underline{b}, \underline{c}] = [\underline{c}, \underline{a}, \underline{b}] = [\underline{b}, \underline{c}, \underline{a}]. \quad (40)$$

We also note that any pairwise swop in the scalar product results in a sign change, that is,

$$[\underline{a}, \underline{b}, \underline{c}] = -[\underline{b}, \underline{a}, \underline{c}] \quad (41)$$

and so on.

This pairwise swop and its associated sign change is easily represented (and reproduced) by a determinant of the components of the three vectors, which correctly gives the volume of the parallelepiped formed by the three vectors. That is, we have

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (42)$$

which, as the property of determinants have it, changes sign under the any pairwise permutation of positions.

6.2. Vector Triplet Product.

The **vector triple product** of *three* vectors \underline{a} , \underline{b} and \underline{c} is defined as the vector that is formed from the product of vector \underline{a} and vector $\underline{b} \times \underline{c}$, that is, the vector triple product is given by

$$\underline{a} \times (\underline{b} \times \underline{c}). \quad (43)$$

A useful identity of the vector triple product is

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}. \quad (44)$$

Before we prove this identity, we note that the vector product $\underline{b} \times \underline{c}$ gives a resultant vector that is normal to both vectors \underline{b} and \underline{c} . When vector \underline{a} crosses with this resultant vector, the final vector has a direction that is normal to both vectors \underline{a} and the resultant

vector, thus, it has a direction that is coplanar with vectors \underline{b} and \underline{c} . Therefore, we note that the vector triple product *can* be written as a linear combination of vectors \underline{b} and \underline{c} , which suggests that Equation (44) is *plausible*.

Again, using Equation (27), we can write the vector triple product as a determinant. We have

$$\underline{a} \times (\underline{b} \times \underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - c_2b_3) & -(b_1c_3 - c_1b_3) & (b_1c_2 - b_2c_1) \end{vmatrix} \quad (45)$$

To prove identity (44), note that for a vector in Cartesian coordinate given by $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$, we can extract the components of vector \underline{r} by considering the dot product of the vector with the basis vector for that component. To obtain the value of scalar b , we simply dot the vector \underline{r} with the basis vector \underline{j} , so that we have

$$\underline{j} \cdot \underline{r} = \underline{j} \cdot (a\underline{i} + b\underline{j} + c\underline{k}) = a(\underline{j} \cdot \underline{i}) + b(\underline{j} \cdot \underline{j}) + c(\underline{j} \cdot \underline{k}) = b$$

where we have used the *orthonormality* of the basis vectors (Equations 2 and 3).

With this in mind, we can easily obtain the x -component of the vector triple product. This is given simply by

$$\begin{aligned} \underline{i} \cdot \underline{a} \times (\underline{b} \times \underline{c}) &= a_2(b_1c_2 - b_2c_1) - a_3[-(b_1c_3 - c_1b_3)] \\ &= a_2c_2b_1 - a_2b_2c_1 + a_3c_3b_1 - a_3b_3c_1 \\ &= a_2c_2b_1 - a_2b_2c_1 + a_3c_3b_1 - a_3b_3c_1 + [a_1c_1b_1 - a_1c_1b_1] \\ &= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3c_3)c_1 \\ &= \underline{i} \cdot [(\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}] \end{aligned} \quad (46)$$

where we introduced extra terms in the third equality.

The y - and z -components of the vector triple product can be similarly proven, giving the final result of Equation (44). The same identity can be proven using the *tensor* method.

This identity of vector triple product allows us to verify that the vector product is **not** associative. Consider

$$\begin{aligned} (\underline{a} \times \underline{b}) \times \underline{c} &= -\underline{c} \times (\underline{a} \times \underline{b}) \\ &= -(\underline{c} \cdot \underline{b})\underline{a} + (\underline{c} \cdot \underline{a})\underline{b} \\ &= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{c} \cdot \underline{b})\underline{a} \\ &\neq (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \\ &= \underline{a} \times (\underline{b} \times \underline{c}) \end{aligned} \quad (47)$$

where the first equality follows from the anti-commutativity of vector product and the second from the identity (Equation 44).

7. VECTOR REPRESENTATION OF LINES AND PLANES

7.1. Equations of Lines.

We can think of a straight line as a collection of points. That is, a line is an example of a **locus**, which is a collection of points satisfying a set of given conditions. In Euclidean 3D space, any point can be represented by a vector having three components; we can write a point as $\underline{r} = (x_0, y_0, z_0)$ wherein the Cartesian basis is assumed. A line is thus a collection of such points so that it can take the vector form of $\underline{r} = (x, y, z)$ with x, y , and z being variables. As they take on different values, the resultant vector gives different points on the straight line.

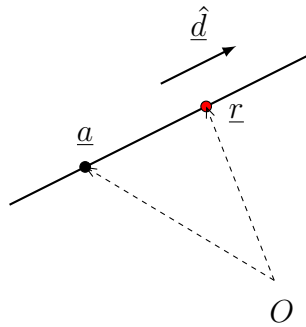


FIGURE 16. A vector representation of a straight line.

To specify a line in vector form, we only need to specify a point on the line and the *direction vector* of the line (Figure 16). Suppose \underline{a} is a point on the line and $\underline{\hat{d}}$ is the unit vector giving the direction of the line, then a *variable* point on the line, represented by \underline{r} , can be obtained using vector addition

$$\underline{r} = \underline{a} + \lambda \underline{\hat{d}} \quad (48)$$

where $|\lambda|$ gives the distance between the points at \underline{a} and \underline{r} . As the value of λ varies, the position vectors \underline{r} traces out the whole straight line.

We can obtain an alternative form of the straight line using cross product. To do so, we can take the cross product of Equation (48) with the direction vector on both sides to obtain

$$\underline{r} \times \underline{\hat{d}} = (\underline{a} + \lambda \underline{\hat{d}}) \times \underline{\hat{d}} = \underline{a} \times \underline{\hat{d}} + \lambda \underline{\hat{d}} \times \underline{\hat{d}} = \underline{a} \times \underline{\hat{d}} \quad (49)$$

where we note that the cross product of any vector with itself returns the zero vector.

We know that any straight line passing through two points is unique, therefore, another way to specify a straight line is using two points. In such case, we can find the direction

of the line using the vectors representing the two points and proceed as above to obtain the equation of the line.

7.1.1. *Perpendicular distance from a point to a line.* Now, consider a point outside a given line (Figure 17). We want to find the perpendicular distance, h , from the point to the line. The geometrical argument gives the distance as

$$h = |\underline{b} - \underline{a}| \sin \theta = |\hat{\underline{d}}| |\underline{b} - \underline{a}| \sin \theta |\hat{\underline{n}}| = |\hat{\underline{d}} \times (\underline{b} - \underline{a})| \quad (50)$$

where we introduced the unit vector in the direction of the line, $\hat{\underline{n}}$, and the unit vector $\hat{\underline{d}}$ that is perpendicular to both vectors $\hat{\underline{d}}$ and $(\underline{b} - \underline{a})$ to satisfy the definition of the cross product.

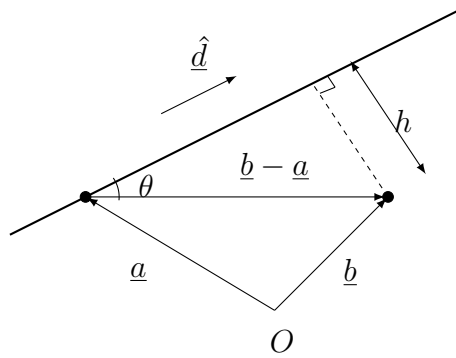


FIGURE 17. Finding the distance from a point to a line using geometrical argument.

7.1.2. *Shortest distance between two skew lines.* Figure 18 shows two skew lines l_1 and l_2 having non-parallel direction vectors $\hat{\underline{l}}$ and $\hat{\underline{m}}$. We wish to find the shortest distance between these two lines, $h = PQ$. The vector equations of these two lines are given by

$$\begin{aligned} l_1 : \quad \underline{r} &= \underline{a} + \lambda \hat{\underline{l}} \\ l_2 : \quad \underline{r} &= \underline{b} + \mu \hat{\underline{m}}. \end{aligned}$$

Since point P is on l_1 , we have $\overrightarrow{OP} = \underline{a} + \lambda \hat{\underline{l}}$ for *some* real λ ; similarly, we have $\overrightarrow{OQ} = \underline{b} + \mu \hat{\underline{m}}$ for *some* real μ . Thus, we have,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \underline{h} = \underline{b} + \mu \hat{\underline{m}} - \underline{a} - \lambda \hat{\underline{l}} \quad (51)$$

Now, \overrightarrow{PQ} is normal to both lines, i.e., perpendicular to *both* direction vectors of the lines, which implies that \overrightarrow{PQ} is parallel to the vector product of the direction vectors, $\hat{\underline{l}} \times \hat{\underline{m}}$ (remember that the normal vector of a vector product is perpendicular to both the

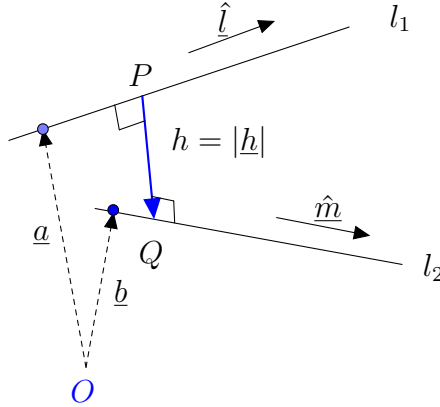


FIGURE 18. Illustration of two skew lines having the shortest distance between them denoted by h .

“crossing” vectors). We can thus take the dot product of Equation (51) to get rid of both unit vectors \hat{l} and \hat{m} in it:

$$\begin{aligned}\overrightarrow{PQ} \cdot (\hat{l} \times \hat{m}) &= (\underline{b} + \mu \hat{m} - \underline{a} - \lambda \hat{l}) \cdot (\hat{l} \times \hat{m}) \\ \underline{h} \cdot (\hat{l} \times \hat{m}) &= (\underline{b} - \underline{a}) \cdot (\hat{l} \times \hat{m})\end{aligned}$$

Noting that the vector \underline{h} is parallel to the vector $\hat{l} \times \hat{m}$, we can expand the left-hand side to $\underline{h} \cdot (\hat{l} \times \hat{m}) = |\underline{h}| |\hat{l} \times \hat{m}| \cos 0 = |\underline{h}| |\hat{l} \times \hat{m}|$. Thus the equation above simplifies to give the shortest distance between two skew lines as

$$h = |\underline{h}| = \left| \frac{(\underline{b} - \underline{a}) \cdot (\hat{l} \times \hat{m})}{|\hat{l} \times \hat{m}|} \right| \quad (52)$$

7.2. Equations of Planes.

Here again, we think of planes as a collection of points in the Euclidean 3D space. A plane is simply another example of a locus. Given any two *non-collinear* vectors, \underline{l} and \underline{m} , we can express any point using a linear combination of these two vectors. That is, the two vectors *span* the two-dimensional space, tracing out a plane. With only these two vectors, the plane is not unique as it can be shifted parallel to both vectors to give a family of planes. If however, the plane is required to additionally pass through a given point, \underline{a} , then it is uniquely defined. Thus, we can define a plane in vector form as

$$\underline{r} = \underline{a} + \lambda \underline{l} + \mu \underline{m} \quad (53)$$

where again, the scalars λ and μ can take all real values so that any point on the plane corresponds to a specific set of λ and μ values.

Figure 19 illustrates this concept. Any point on a plane that passes through a point A with two given direction vectors \underline{l} and \underline{m} , can be represented by a position vector \underline{r} . To obtain this position vector $\underline{r} = \overrightarrow{OR}$, we add vectorially so that $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BR}$, giving the definition of the plane in Equation (53).

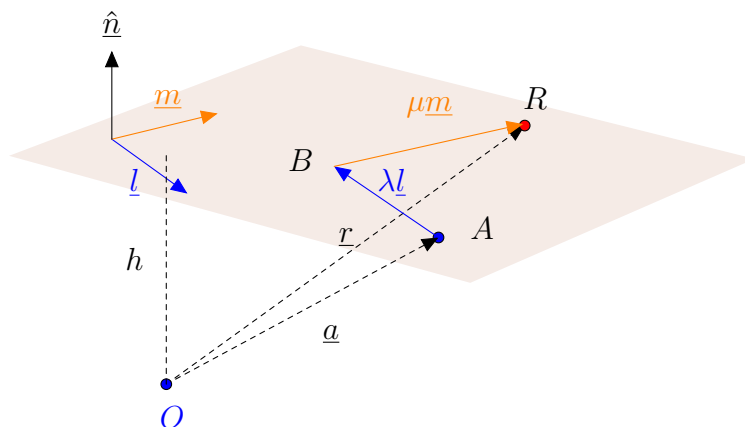


FIGURE 19. Equation of a plane as a collection of points with position vector \underline{r} .

We can obtain an alternative form of the vector equation of a plane by noting that the normal vector to the plane is simply given by the cross product of vectors \underline{l} and \underline{m} , that is, $\hat{n} \propto \underline{l} \times \underline{m}$. We could then take the dot product of Equation (53) with \hat{n} on both sides to obtain

$$\begin{aligned}
 \underline{r} \cdot \hat{n} &= (\underline{a} + \lambda \underline{l} + \mu \underline{m}) \cdot \hat{n} \\
 &= \underline{a} \cdot \hat{n} + \lambda \underline{l} \cdot \hat{n} + \mu \underline{m} \cdot \hat{n} \\
 &= \underline{a} \cdot \hat{n} \\
 &= h
 \end{aligned}$$

where the second last equality follows since vector \hat{n} is normal to both vectors \underline{l} and \underline{m} and the last follows since $\underline{a} \cdot \hat{n}$ gives the length of projection of vector \underline{a} in the direction of the vector \hat{n} , which gives the perpendicular distance, h , from the origin to the plane (Figure 19). Therefore, we have

$$\underline{r} \cdot \hat{n} = \underline{a} \cdot \hat{n} = h \quad (54)$$

We also note that three points specify a unique plane. We can form two direction vectors using any two points from the given three and together with any one point, we will be able to find the equation passing through these three points using the procedure above.

7.2.1. *Distance from a point to a plane.* Here, we want to find the perpendicular distance from a point A whose position vector is \underline{a} to a plane with an equation

$$\underline{r} \cdot \hat{n} = h \quad (55)$$

(Figure 20). We note that the line passing through line segment AB has the same direction as the normal vector of the plane \hat{n} . Therefore, we can write the equation of

this line as

$$\underline{r} = \underline{a} + \lambda \hat{n}$$

for *all* real values of λ .

Since the point B lies on this plane, its position vector is given by

$$\underline{r} = \underline{a} + \lambda_0 \hat{n} = \underline{a} + k \hat{n} \quad (56)$$

for *some* real value λ_0 . Note that $\lambda_0 = k$, giving the distance from point A to point B .

Now, point B also lies on the given plane, thus, its position vector satisfies the equation of the plane as well. Substituting Equation (56) to Equation (55), we obtain

$$\begin{aligned} (\underline{a} + k \hat{n}) \cdot \hat{n} &= h \\ \implies \underline{a} \cdot \hat{n} + k &= h \\ \implies k &= |h - \underline{a} \cdot \hat{n}| \end{aligned} \quad (57)$$

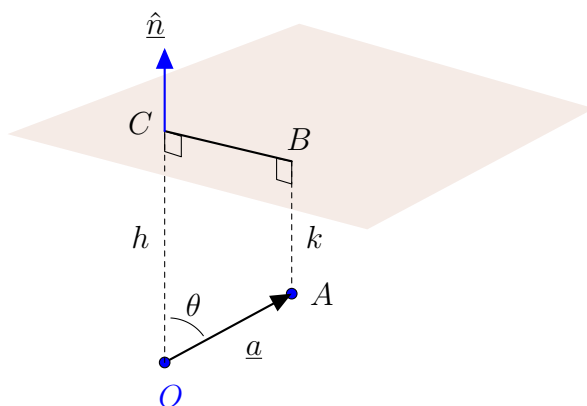


FIGURE 20. Distance from a point given by position vector \underline{a} to a given plane.

The result above simply says that distance k is obtained by subtracting the length of projection of vector \underline{a} onto the normal vector of the plane \hat{n} from distance h . This is simply

$$k = |h - |\underline{a}| \cos \theta| \quad (58)$$

which follows directly from geometrical argument.

7.2.2. *Common point on three given planes.* Given the vector equations of three planes

$$\Pi_1 : \quad \underline{r} \cdot \underline{a} = l \quad (59)$$

$$\Pi_2 : \quad \underline{r} \cdot \underline{b} = m \quad (60)$$

$$\Pi_3 : \quad \underline{r} \cdot \underline{c} = n \quad (61)$$

where \underline{a} , \underline{b} and \underline{c} are the normal vectors of the three planes respectively. We wish to find the point (if any) of the three intersecting planes. Algebraically, we could solve the three equations with three unknowns simultaneously. However, we will consider solving this using vector algebra.

We consider any two planes first. Suppose we consider planes Π_1 and Π_2 . These two planes intersect at a common line whose direction vector is perpendicular to the normal vectors of both planes, that is the direction vector of the resulting line is parallel to $\underline{a} \times \underline{b}$. Using Equation (49), we can write the vector equation of the intersecting line as $\underline{r} \times (\underline{a} \times \underline{b})$ which expands, using the vector triple product identity (Equation 44), to

$$\begin{aligned} \underline{r} \times (\underline{a} \times \underline{b}) &= (\underline{r} \cdot \underline{b})\underline{a} - (\underline{r} \cdot \underline{a})\underline{b} \\ &= m\underline{a} - l\underline{b} \end{aligned} \tag{62}$$

We note that any point that results from the intersecting planes will not only satisfy the equation of the intersecting line above, but also the equation of the plane Π_3 . This implies that the position vector \underline{r} that satisfies Equations (61) and (62) is the point intersected by *all* three planes. To proceed, we take the vector product of Equation (62) with the normal vector \underline{c} to obtain

$$\begin{aligned} m(\underline{c} \times \underline{a}) - l(\underline{c} \times \underline{b}) &= \underline{c} \times [\underline{r} \times (\underline{a} \times \underline{b})] \\ &= \underline{c} \cdot (\underline{a} \times \underline{b})\underline{r} - \underline{c} \cdot \underline{r}(\underline{a} \times \underline{b}) \\ &= [\underline{c}, \underline{a}, \underline{b}]\underline{r} - n(\underline{a} \times \underline{b}) \\ &= [\underline{a}, \underline{b}, \underline{c}]\underline{r} - n(\underline{a} \times \underline{b}) \\ \implies \underline{r} &= \frac{l(\underline{b} \times \underline{c}) + m(\underline{c} \times \underline{a}) + n(\underline{a} \times \underline{b})}{[\underline{a}, \underline{b}, \underline{c}]} \end{aligned} \tag{63}$$

We can easily check that this point satisfies the equations of the planes (59), (60) and (61). It is also evident that the condition for there to be a point of intersection of three given planes is that $[\underline{a}, \underline{b}, \underline{c}] \neq 0$, which requires the normal vectors of the three planes to be non-coplanar.

THE END